Rabies in Raccoons: Optimal Control for a Discrete Time Model on a Spatial Grid

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Outline

- Background
- Assumptions and format of the model
- Optimal control formulation and analysis
- Numerical results
- Conclusion

Rabies in Raccoons

- Rabies is a common viral disease.
- Transmission through the bite of an infected animal.
- Raccoons are the primary vector for rabies in eastern US.
- Vaccine is distributed through food baits.

http://www.cdc.gov

Reported Cases of Rabies, 2001

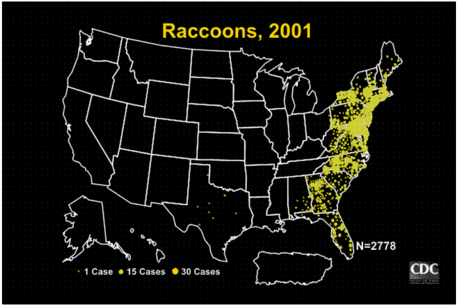


Figure: Reported Cases of Rabies, 2001, *http://www.cdc.gov*

Basic Assumptions

The objective of the problem formulation is to provide a simple, readily modified framework to analyze spatial optimal control for vaccine distribution as it impacts the spread of rabies among raccoons.

The epidemiological assumptions:

- No variance in time from infection to death
- Random mixing assumed to be the only means of contact and transmission

Temporal Set-up

- Time scale: There is no population growth or immigration in the model presented here, but is included in a more general model. The scale is assumed to be over a time period (say within a season) over which births do not occur.
- Mortality occurs only due to infection.
- The time step of each iteration is that over which all infected raccoons die (e.g. about 10 days).

Spatial set-up

- Spatial scale: each cell is uniform in size, arranged rectangularly
- Movement: Raccoons are assumed to move according to a movement matrix from cell to cell, with distance dependence in dispersal.

Vaccine

Vaccine/food packets are assumed to be reduced each time step due to uptake by raccoons, with the remaining packets then decaying due to other factors.

Then additional packets (CONTROL variable) are added at the end of each time step.

Variables

Model with (k,l) denoting spatial location, t time

- susceptibles = S(k,l,t)
- infecteds = I(k,l,t)
- immune = R(k,l,t)
- vaccine = v(k,l,t)
- control c(k, l, t), input of vaccine baits

Order of events

Within a time step (about a week to 10 days):

- First movement: using home range estimate to get range of movement. See sum S, sum I and sum R to reflect movement.
- Then: some susceptibles become immune by interacting with vaccine
- Lastly: new infecteds from the interaction of the non-immune susceptibles and infecteds

NOTE that infecteds from time step n die and do not appear in time step $n + 1$.

Susceptibles and Infecteds Equations

$$S(k,l,t+1) = \left(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K}\right)\text{sum\_}S(k,l,t)$$
$$- \beta \frac{\left(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K}\right)\text{sum\_}S(k,l,t)\text{sum\_}I(k,l,t)}{\text{sum\_}S(k,l,t) + \text{sum\_}R(k,l,t) + \text{sum\_}I(k,l,t)},$$

Susceptibles and Infecteds Equations

$$S(k,l,t+1) = \left(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K}\right)\text{sum\_}S(k,l,t) - \beta \frac{\left(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K}\right)\text{sum\_}S(k,l,t)\,\text{sum\_}I(k,l,t)}{\text{sum\_}S(k,l,t)+\text{sum\_}R(k,l,t)+\text{sum\_}I(k,l,t)},$$

$$I(k,l,t+1) = \beta \frac{\left(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K}\right)\text{sum\_}S(k,l,t)\,\text{sum\_}I(k,l,t)}{\text{sum\_}S(k,l,t)+\text{sum\_}R(k,l,t)+\text{sum\_}I(k,l,t)}.$$

Immune and Vaccine Equations

$$R(k,l,t+1) = \text{sum\_}R(k,l,t) + e_1 \frac{v(k,l,t)}{v(k,l,t)+K}\text{sum\_}S(k,l,t),$$

$$v(k,l,t+1) =$$
$$Dv(k,l,t)\ \max\left[0, (1 - e(\text{sum\_}S(k,l,t) + \text{sum\_}R(k,l,t)))\right] + c(k,l,t).$$

States and Control

- States: S(m, n, t), I(m, n, t), R(m, n, t), v(m, n, t) for $t = 2, \ldots T$ (given initial distribution at $t = 1$)
- Control c(m, n, t) , $t = 1, 2, \ldots, T\text{-}1$

Objective Functional

maximize the susceptible raccoons, minimize the infecteds and cost of distributing baits

$$\sum_{m,n} \Big(I(m,n,T) - S(m,n,T) \Big) + B \sum_{m,n,t} c(m,n,t)^2,$$

where T is the final time and $c(m,n,t)$ is the cost of distributing the packets at cell (m,n) and time t, B is the balancing coefficient, c is the control.

Use discrete version of Pontryagin's Maximum Principle.

Hamiltonian at time t

$$
\begin{aligned}
H(m,n,t) = & B \sum_{m,n} c(m,n,t)^2 \\
& + \sum_{m,n} \Big[LS(m,n,t+1)\Big(\text{RHS of } S(m,n,t+1) \text{ eqn}\Big) \\
& + LI(m,n,t+1)\Big(\text{RHS of } I(m,n,t+1) \text{ eqn}\Big) \\
& + LR(m,n,t+1)\Big(\text{RHS of } R(m,n,t+1) \text{ eqn}\Big) \\
& + Lv(m,n,t+1)\Big(\text{RHS of } v(m,n,t+1) \text{ eqn}\Big) \Big]
\end{aligned}
$$

Adjoints and Optimal Control

LS, LI, LR, Lv denote the adjoints for S, I, R, v respectively

$$LS(i,j,t) = \frac{\partial H(t)}{\partial S(i,j,t)},$$

$$\frac{\partial H(t)}{\partial c(i,j,t)} = 2Bc(i,j,t) + Lv(i,j,t+1) = 0.$$

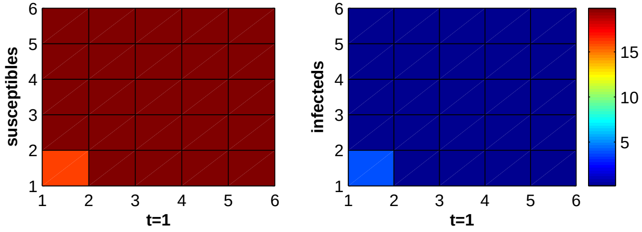
$$\Longrightarrow c^*(i,j,t) = -\frac{1}{2B} Lv(i,j,t+1),$$

subject to the upper and lower bounds

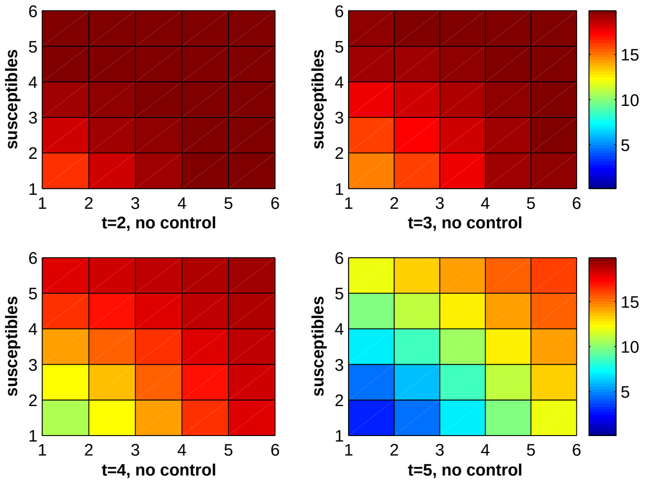
Numerical Iterative Method

- Start with a control guess and initial distribution of raccoons
- Solve the state equations forward
- Solve the adjoint equations backwards, using LI(k, l, T)=1, LS(k,l, T) =-1, other adjoints are zero at final time
- Update the control using the characterization
- Repeat until convergence

Disease Starts From the Corner: Initial Distribution



Susceptibles, no control



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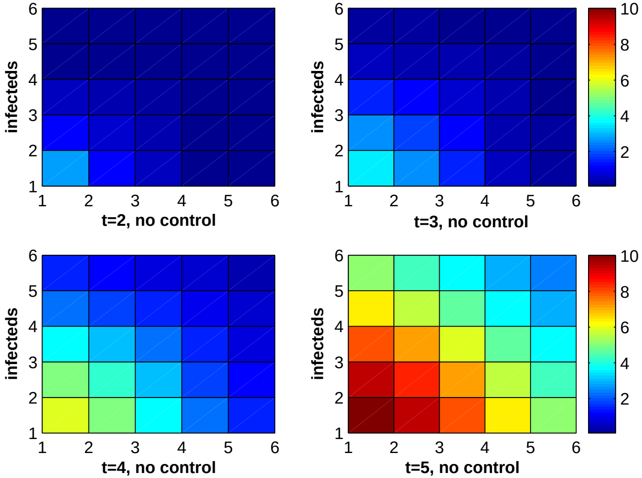
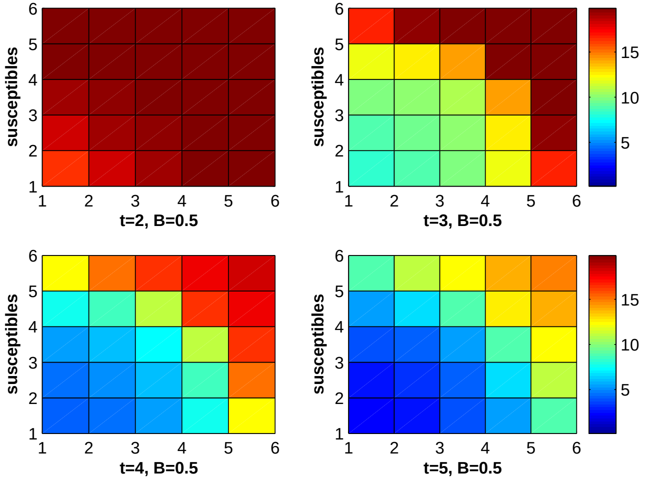


Susceptibles, with control, $B = 0.5$



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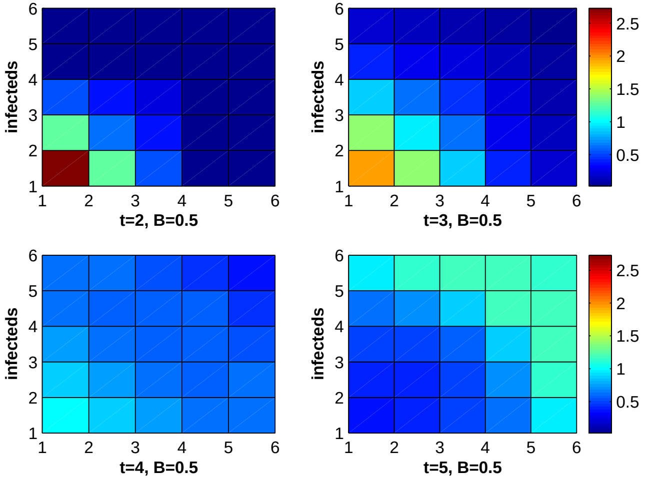
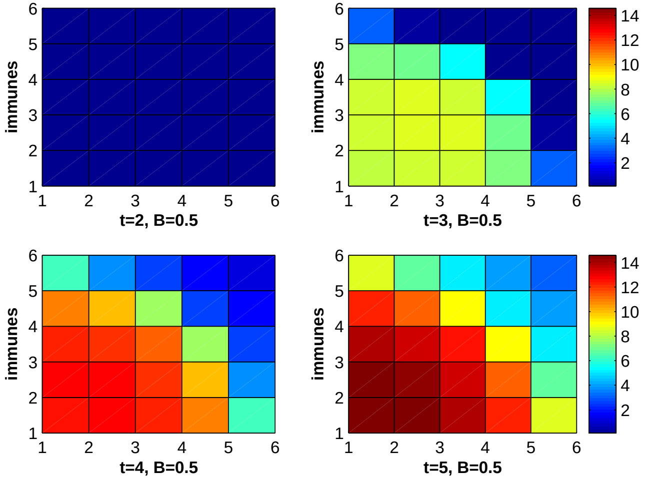
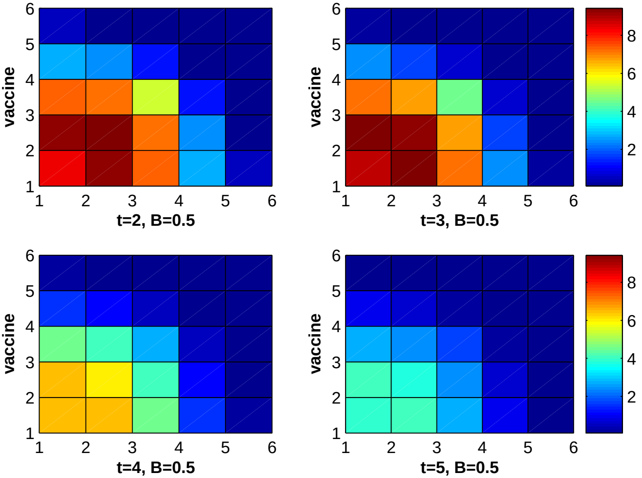


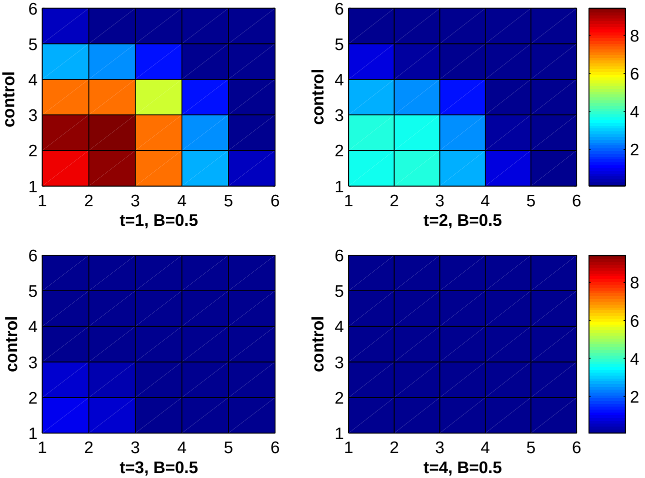
Immune, with control, $B = 0.5$



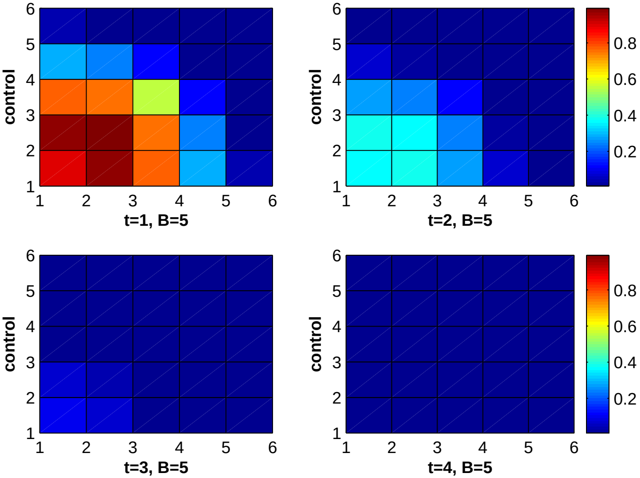
Vaccine, with control, $B = 0.5$



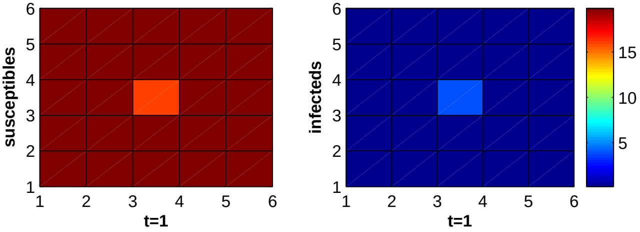
Optimal Control, $B = 0.5$



Optimal Control, $B = 5$



Disease Starts From the Center: Initial Distribution



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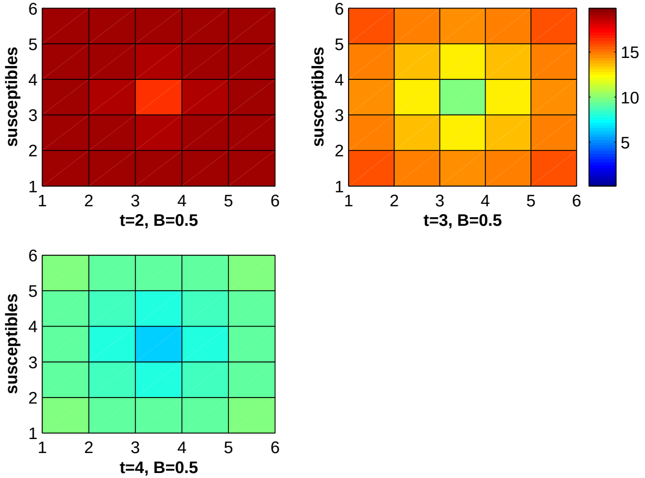

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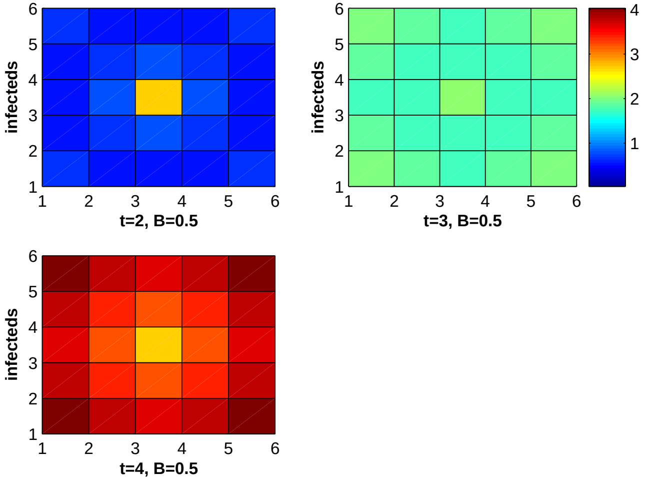


Optimal Control, $B = 0.5$, $t = 1$

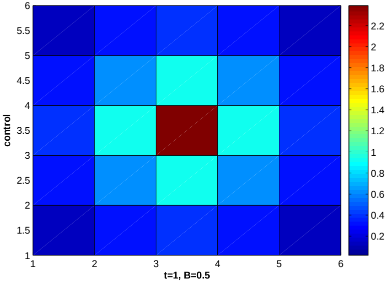
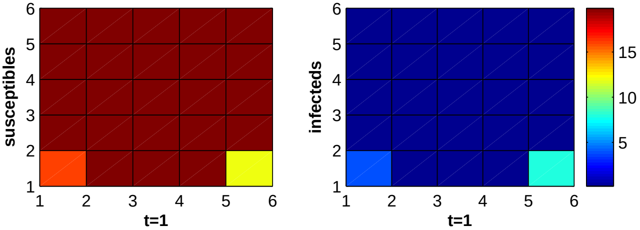


Figure: Optimal Control, $B = 0.5$, $t = 1$

Inhomogeneous Initial Distribution



Optimal Control, $B = 0.5$, $t = 1$

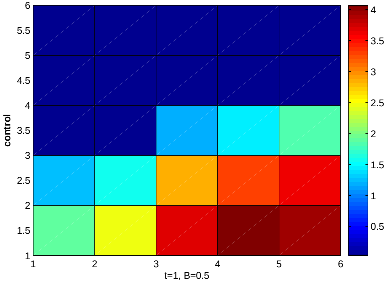


Figure: Optimal Control, $B = 0.5$, $t = 1$

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- Developed a method and model to determine different optimal distributions of vaccine to control rabies spread;
- Illustrated the approach using three scenarios;
- Optimal bait distribution depends on the initial location of the disease outbreak and the distribution of raccoons throughout the grid;
- The method can be readily extended to evaluate optimal vaccination distribution strategies with other spatially heterogeneous interactions, larger spatial grids and different movement assumptions.