Mathematical Analysis of Epidemiological Models II

Jan Medlock

Clemson University
Department of Mathematical Sciences

22 July 2009

Endemic Model

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu N - \beta \frac{I(t)}{N} S(t) - \mu S(t)$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta \frac{I(t)}{N} S(t) - \gamma I(t) - \mu I(t)$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I(t) - \mu R(t)$$

Constant population size:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\mathrm{d}R}{\mathrm{d}t} = 0.$$

Endemic Model

So divide by population size

$$s = \frac{S}{N}, \quad i = \frac{I}{N}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{S}{N}\right) = \frac{1}{N}\frac{\mathrm{d}S}{\mathrm{d}t} = \mu\frac{N}{N} - \beta\frac{I}{N}\frac{S}{N} - \mu\frac{S}{N} = \mu - \beta is - \mu s$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \beta is - \gamma i - \mu i$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \gamma i - \mu r$$

$$s + i + r = 1$$

Find equilibria

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}i}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0 = \mu - \beta is - \mu s$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = 0 = \beta is - \gamma i - \mu i$$

Two equilibria:

- Disease-free equilibrium:

$$E_0 = (s = 1, i = 0)$$

- Endemic equilibrium:

$$E_e = \left(s = \frac{\gamma + \mu}{\beta}, i = \frac{\mu(\beta - \gamma - \mu)}{\beta(\gamma + \mu)}\right)$$

Linearize equations

Write as vector differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} s \\ i \end{pmatrix} = \begin{pmatrix} \mu - \beta i s - \mu s \\ \beta i s - \gamma i - \mu i \end{pmatrix} = \mathbf{f}(s, i)$$

By Taylor's theorem

$$\mathbf{f}(s, i) = \mathbf{f}(s_0, i_0) + \mathbf{J}(s_0, i_0)\left[\begin{pmatrix} s \\ i \end{pmatrix} - \begin{pmatrix} s_0 \\ i_0 \end{pmatrix}\right] + \cdots$$

At equilibrium, $\mathbf{f}(s_0, i_0) = \mathbf{0}$, so the dynamics near (s_0, i_0) are governed by the linear part $\mathbf{J}(s_0, i_0)$

Analysis

Jacobian derivative of **f**

$$\mathbf{J}(s,i) = \begin{bmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial i} \\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial i} \end{bmatrix} = \begin{bmatrix} -\beta i - \mu & -\beta s \\ \beta i & \beta s - \gamma - \mu \end{bmatrix}$$

- Disease-free equilibrium

$$\mathbf{J}(1,0) = \begin{bmatrix} -\mu & -\beta \\ 0 & \beta - \gamma - \mu \end{bmatrix}$$

Eigenvalues $\{-\mu, \beta - \mu - \gamma\}$

  - $\lambda_1 = -\mu < 0$
  - $\lambda_2 = \beta - \mu - \gamma$
    - $\beta - \mu - \gamma < 0 \iff \frac{\beta}{\gamma+\mu} < 1$, stable, No epidemic
    - $\beta - \mu - \gamma > 0 \iff \frac{\beta}{\gamma+\mu} > 1$, unstable, Epidemic

$$R_0 = \frac{\beta}{\gamma + \mu}$$

Analysis

- Endemic equilibrium

$$\mathbf{J}\left(\frac{\gamma+\mu}{\beta}, \frac{\mu(\beta-\gamma-\mu)}{\beta(\gamma+\mu)}\right) = \begin{bmatrix} -\frac{\mu\beta}{\gamma+\mu} & -\gamma-\mu \\ \frac{\mu(\beta-\gamma-\mu)}{\gamma+\mu} & 0 \end{bmatrix}$$

Eigenvalues $\left\{-\frac{\mu\beta}{\mu+\gamma} \pm \sqrt{\frac{\mu^2\beta^2}{(\mu+\gamma)^2} - 4\mu(\beta-\gamma-\mu)}\right\}$

  - $R_0 = \frac{\beta}{\gamma+\mu} > 1$, stable
  - $R_0 = \frac{\beta}{\gamma+\mu} < 1$, unstable

Summary

$$R_0 = \frac{\beta}{\gamma + \mu}$$

$$E_0 = (s = 1, i = 0)$$

$$E_e = \left(s = \frac{\gamma + \mu}{\beta} = \frac{1}{R_0}, i = \frac{\mu(\beta - \gamma - \mu)}{\beta(\gamma + \mu)} = \frac{\mu}{\beta}(1 - R_0) \right)$$

- $R_0 < 1$
  Disease-free equilibrium is stable
  Endemic equilibrium is unstable (and nonsense!)
- $R_0 > 1$
  Disease-free equilibrium is unstable
  Endemic equilibrium is stable

Vaccination model

$$\frac{\mathrm{d}s}{\mathrm{d}t} = (1-p)\mu - \beta is - \mu s$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = \beta is - \gamma i - \mu i$$
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \gamma i - \mu r$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = p\mu - \mu v$$

$$s + i + r + v = 1$$

Analysis

Disease-free equilibrium:

$$E_0 = (s = 1 - p, i = 0, v = p)$$

Jacobian:

$$\mathbf{J}(s, i, v) = \begin{bmatrix} -\beta i - \mu & -\beta s & 0 \\ \beta i & \beta s - \gamma - \mu & 0 \\ 0 & 0 & -\mu \end{bmatrix}$$

$$\mathbf{J}(E_0) = \begin{bmatrix} -\mu & -\beta(1-p) & 0 \\ 0 & \beta(1-p) - \gamma - \mu & 0 \\ 0 & 0 & -\mu \end{bmatrix}$$

Analysis

$$\mathbf{J}(E_0) = \begin{bmatrix} -\mu & -\beta(1-p) & 0 \\ 0 & \beta(1-p)-\gamma-\mu & 0 \\ 0 & 0 & -\mu \end{bmatrix}$$

$$\lambda_{1,2} = -\mu < 0$$

$$\lambda_3 = \beta(1-p) - \gamma - \mu$$

$$\lambda_3 > 0 \iff R_v = \frac{\beta}{\gamma+\mu}(1-p) = R_0(1-p) > 1$$

$$\lambda_3 < 0 \iff R_v < 1$$

Stability determined by R_v

Critical vaccination level

$$R_v = R_0(1 - p^*) = 1 \implies p^* = 1 - \frac{1}{R_0}$$

$$p > p^* \implies R_v < 1$$ No epidemic!

Tomorrow

- R_0 for complex models
- Vector-borne disease model
- Age-structured model