# Mathematical Analysis of Epidemiological Models I

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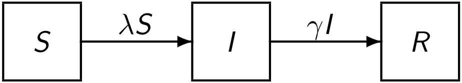
# The General Epidemic, or SIR, Model

- We divide the population into three groups:
  - Susceptible individuals, $S(t)$
  - Infective individuals, $I(t)$
  - Recovered individuals, $R(t)$




- Assumptions
  - Population size is large and constant, $S(t) + I(t) + R(t) = N$
  - No birth, death, immigration or emigration
  - No latent period
  - Homogeneous mixing
  - Infection rate is proportional to the fraction of infectives (frequency dependent), i.e. $\lambda = \beta I/N$
  - Recovery rate is constant, $\gamma$

# Model Equations

- A system of three ordinary differential equations describes this model:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta\frac{I(t)}{N}S(t)$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta\frac{I(t)}{N}S(t) - \gamma I(t)$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I(t)$$

or

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta\frac{I(t)}{N}S(t)$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta\frac{I(t)}{N}S(t) - \gamma I(t)$$
$$R(t) = N - S(t) - I(t)$$

# $R_0$

- How many new infectives are caused by a single infective introduced into a population that is entirely susceptible?
- In this case the second ODE is

$$\frac{\mathrm{d}I}{\mathrm{d}t} \approx (\beta - \gamma) I(t).$$

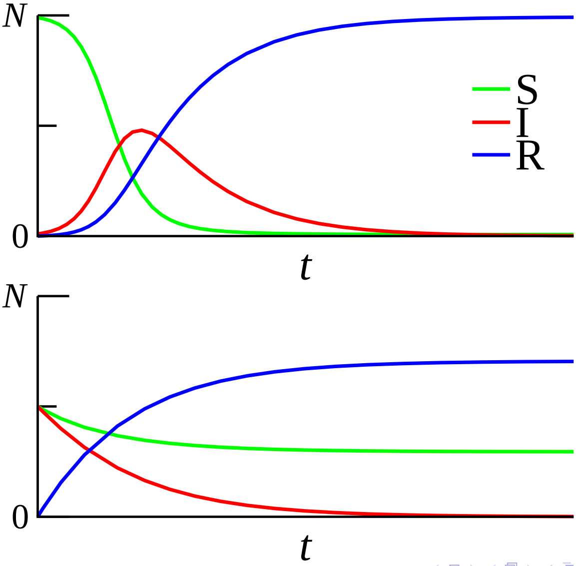
So if $\beta - \gamma > 0$ then $I(t)$ increases.

- Define the basic reproductive number

$$R_0 = \frac{\beta}{\gamma}.$$

If $R_0 > 1$ then $I(t)$ increases and we have an epidemic.

# Epidemic Curves



# Linear Stability Analysis

Step 1 Find equilibrium points

Step 2 Linearize at each point

Step 3 Find eigenvalues of linearized problem

# Find equilibrium points

Equilibria are points where the variables do not change with time: i.e. $\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t} = 0$.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0 = -\beta \frac{I}{N} S$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0 = \beta \frac{I}{N} S - \gamma I$$

This gives two equilibria:

$$(S = N, I = 0) \text{ and } (S = 0, I = 0)$$

We'll concentrate on the first, the disease-free equilibrium. (The second is after an epidemic, when $R = N$.)

# Linearize equations

First, we'll shift the variables so that the origin is at the equilibrium $(N, 0) \to (0, 0)$:

$$S^* = N - S$$
$$I^* = I - 0 = I$$

Then

$$\frac{\mathrm{d}S^*}{\mathrm{d}t} = \frac{\mathrm{d}N}{\mathrm{d}t} - \frac{\mathrm{d}S}{\mathrm{d}t} = \beta\frac{I}{N}S = \beta\frac{I^*}{N}(N - S^*)$$
$$\frac{\mathrm{d}I^*}{\mathrm{d}t} = \frac{\mathrm{d}I}{\mathrm{d}t} = \beta\frac{I}{N}S - \gamma I = \beta\frac{I^*}{N}(N - S^*) - \gamma I^*$$

# Linearize equations

We are only going to consider small deviations from the equilibrium, so that $S^*$ and $I^*$ are small. That means that any terms with higher powers of $S^*$ and $I^*$ are very small, so we neglect them. This is linearization.

$$\frac{\mathrm{d}S^*}{\mathrm{d}t} = \beta \frac{I^*}{N}(N - S^*) = \beta I^* - \beta \frac{I^* S^*}{N} \to \beta I^*$$
$$\frac{\mathrm{d}I^*}{\mathrm{d}t} = \beta \frac{I^*}{N}(N - S^*) - \gamma I^* = \beta I^* - \beta \frac{I^* S^*}{N} - \gamma I^* \to \beta I^* - \gamma I^*$$

# Linearize equations

Our linearized equations are

$$\frac{\mathrm{d}S^*}{\mathrm{d}t} = \beta I^*$$
$$\frac{\mathrm{d}I^*}{\mathrm{d}t} = \beta I^* - \gamma I^*$$

(Autonomous) linear differential equations have solutions $\mathbf{v}\mathrm{e}^{rt}$.
Using that form of solution, without yet knowing $r$, we get

$$\frac{\mathrm{d}S^*}{\mathrm{d}t} = \frac{\mathrm{d}S_0\mathrm{e}^{rt}}{\mathrm{d}t} = S_0 r\mathrm{e}^{rt} = \beta I_0\mathrm{e}^{rt}$$
$$\frac{\mathrm{d}I^*}{\mathrm{d}t} = \frac{\mathrm{d}I_0\mathrm{e}^{rt}}{\mathrm{d}t} = I_0 r\mathrm{e}^{rt} = (\beta - \gamma)I_0\mathrm{e}^{rt}$$

# Linearize equations

Dividing by $\mathrm{e}^{rt}$, which is never 0, gives the linear algebra problem

$$rS_0 = \beta I_0$$
$$rI_0 = (\beta - \gamma) I_0$$

Transforming that into a standard vector–matrix problem gives the standard eigenvalue problem

$$r \begin{pmatrix} S_0 \\ I_0 \end{pmatrix} = \begin{bmatrix} 0 & \beta \\ 0 & \beta - \gamma \end{bmatrix} \begin{pmatrix} S_0 \\ I_0 \end{pmatrix} \implies \begin{bmatrix} r & -\beta \\ 0 & r - \beta + \gamma \end{bmatrix} \begin{pmatrix} S_0 \\ I_0 \end{pmatrix} = \mathbf{0}$$

# Linearize equations

This is solved by setting the determinant of the matrix to 0:

$$\det\left(\begin{bmatrix} r & -\beta \\ 0 & r-\beta+\gamma \end{bmatrix}\right) = r(r-\beta+\gamma)-(-\beta)0 = r(r-\beta+\gamma) = 0$$

This gives the two eigenvalues

$$r = 0 \text{ and } r = \beta - \gamma.$$

(In general, for an $N \times N$ matrix, we get $N$ eigenvalues.)
Here, the eigenvalue of interest is $r = \beta - \gamma$.

# Linearize equations

With the eigenvalue $r = \beta - \gamma$, we have solutions with $\mathrm{e}^{rt}$.

- If $r = \beta - \gamma > 0$, the solutions grow away from the equilibrium. The equilibrium is unstable. For our model, this is an epidemic.
- If $r = \beta - \gamma < 0$, the solutions contract back towards the equilibrium. The equilibrium is stable. For our model, this is no epidemic.

# Linearize equations — The quick way

If we first write the model as a vector differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} -\beta \frac{I(t)}{N} S(t) \\ \beta \frac{I(t)}{N} S(t) - \gamma I(t) \end{pmatrix}$$

Generically, this is in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \mathbf{f}(\mathbf{v})$$

# Linearize equations — The quick way

Taylor's theorem for a scalar-valued function of a scalar ($f : \mathbb{R} \to \mathbb{R}$) with enough smoothness

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) + \cdots$$

This is also true for a vector-valued function of a vector ($\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$)

$$\mathbf{f}(\mathbf{v}) = \mathbf{f}(\mathbf{v}_0) + \mathbf{J}(\mathbf{f}, \mathbf{v}_0)(\mathbf{v} - \mathbf{v}_0) + \cdots$$

where $\mathbf{J}(\mathbf{f}, \mathbf{v}_0)$ is the Jacobian of $\mathbf{f}$, the generalization of the first derivative $f'(x_0)$.
The Taylor expansion explicitly separates the function into its constant part, its linear part, etc.

# Linearize equations — The quick way

Then, letting $\mathbf{v}_0 = (N, 0)$, the equilibrium

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} S \\ I \end{pmatrix} &= \begin{pmatrix} -\beta \frac{I}{N} S \\ \beta \frac{I}{N} S - \gamma I \end{pmatrix} = \begin{pmatrix} -\beta \frac{I}{N} S \\ \beta \frac{I}{N} S - \gamma I \end{pmatrix}_{S=N, I=0} \\
&\quad + \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}S}\left(-\beta \frac{I}{N} S\right) & \frac{\mathrm{d}}{\mathrm{d}I}\left(-\beta \frac{I}{N} S\right)) \\ \frac{\mathrm{d}}{\mathrm{d}S}\left(\beta \frac{I}{N} S - \gamma I\right) & \frac{\mathrm{d}}{\mathrm{d}I}\left(\beta \frac{I}{N} S - \gamma I\right) \end{bmatrix}_{S=N, I=0} (\mathbf{v} - \mathbf{v}_0) + \cdots \\
&= \mathbf{0} + \begin{bmatrix} -\beta \frac{I}{N} & -\beta \frac{S}{N} \\ \beta \frac{I}{N} & \beta \frac{S}{N} - \gamma \end{bmatrix}_{S=N, I=0} (\mathbf{v} - \mathbf{v}_0) + \cdots \\
&= \begin{bmatrix} 0 & -\beta \\ 0 & \beta - \gamma \end{bmatrix} (\mathbf{v} - \mathbf{v}_0) + \cdots
\end{aligned}
$$

# Linearize equations — The quick way

First, note that the constant part of the Taylor expansion, $\mathbf{f}(\mathbf{v}_0) = \mathbf{0}$ when $\mathbf{v}_0$ is an equilibrium: this is how we found the equilibria!

Following the same steps as before, we see that ultimately, all we need the eigenvalues of the Jacobian matrix

$$\begin{bmatrix} 0 & -\beta \\ 0 & \beta - \gamma \end{bmatrix}$$

As before, the eigenvalues are 0 and $\beta - \gamma$.

# Hartman–Grobman Theorem

All of this is backed up formally by the Hartman–Grobman Theorem, which roughly says that near an equilibrium point, the dynamics of the original (nonlinear) system are the same as those for the linearized system.

(This theorem requires that none of the eigenvalues have 0 real part. The model we looked at did not satisfy this condition because one of the eigenvalues was 0, but we could show that this theorem is true nonetheless.)