Optimal Control of Discrete Models

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Introduction

For many populations, births and growth occur in regular times each year (or each cycle). Discrete time models are well suited to describe the life histories of organisms with discrete reproduction and/or growth.

Another example

For example, the Beverton-Holt stock-recruitment model for a population N_t at time t is

$$N_{t+1} = rN_t \left(1 + N_t \frac{r-1}{K}\right)^{-1}.$$

Age structured models

A population may be divided into separate discrete age classes.
At each time step, a certain proportion of each class may survive and enter the next age class.
Individuals in the first age class originate through reproduction from other classes.

$$N_1(t+1) = f_1 N_1(t) + f_2 N_2(t) + f_3 N_3(t)$$

$$N_2(t+l) = p_1 N_1(t)$$

$$N_3(t+1) = p_2 N_2(t)$$

Simple Control Problem

$$\min_u \sum_{k=0}^{2} \frac{1}{2}\left[x_k^2 + Bu_k^2\right]$$

subject to $\quad x_{k+1} = x_k + u_k \qquad$ for $k = 0, 1, 2,$

$$x_0 = 5.$$

state x_0, x_1, x_2, x_3
control u_0, u_1, u_2, one less component

Control Problem

Given a control $u = (u_0, u_1, \ldots, u_{T-1})$ and initial state x_0, the state equation is given by the difference equation

$$x_{k+1} = g(x_k, u_k, k)$$

for $k = 0, 1, 2, \ldots, T-1$. Note that the state has one more component than the control

$$x = (x_0, x_1, \ldots, x_T),$$

Goal

$$J(u) = \phi(x_T) + \sum_{k=0}^{T-1} f(x_k, u_k, k)$$

Hamiltonian

$$H_k = f(x_k, u_k, k) + \lambda_{k+1} g(x_k, u_k, k), \quad \text{for } k = 0, 1, \ldots, T$$

Notice the indexing on the adjoint.
Necessary conditions

$$\lambda_k = \frac{\partial H_k}{\partial x_k}$$

$$\lambda_T = \phi'(x_T^*)$$

$$\frac{\partial H_k}{\partial u_k} = 0 \text{ at } u^*.$$

Simple Control Problem

$$\min_u \sum_{k=0}^{2} \frac{1}{2}\left[x_k^2 + u_k^2\right]$$

subject to $\quad x_{k+1} = x_k + u_k \quad$ for $k = 0, 1, 2,$

$$x_0 = 5.$$

The control is the input (growth or decay).
What optimal control is expected? (min state and size of control)

Optimality Conditions

$$H_k = \frac{1}{2}\left[x_k^2 + u_k^2\right] + \lambda_{k+1}(x_k + u_k).$$

Our necessary conditions are

$$\lambda_k = \frac{\partial H_k}{\partial x_k} = x_k + \lambda_{k+1} \quad \text{for } k = 0, 1, 2,$$

$$\lambda_3 = 0,$$

$$0 = \frac{\partial H_k}{\partial u_k} = u_k + \lambda_{k+1} \quad \text{at } u_k^*.$$

continuing

$$x_{k+1} = x_k - \lambda_{k+1} \qquad \text{for } k = 0, 1, 2.$$

$$\lambda_3 = 0$$

$$x_3 = x_2 \qquad \text{and} \qquad \lambda_2 = x_2.$$

$$x_2 = x_1 - \lambda_2$$

solve algebraic equations

Control Answers

optimal state values

$x_1^* = 2, x_2^* = 1, x_3^* = 1$

optimal control values

$u_0^* = -3, u_1^* = -1, u_2^* = 0.$

System Case

$$x_{j,k+1} = g_j(x_{1,k}, \ldots, x_{n,k}, u_{1,k}, \ldots, u_{m,k}, k)$$

for $k = 0, 1, 2, \ldots, T-1$, $j = 1, 2, \ldots, n$.

$$x_j = (x_{j,0}, x_{j,1}, \ldots, x_{j,T}).$$

There are m controls, n states, and T time steps. Define the objective functional as

$$J(u) = \phi(x_{1,T}, \ldots, x_{n,T}) +$$

$$\sum_{k=0}^{T-1} f(x_{1,k}, \ldots, x_{n,k}, u_{1,k}, \ldots, u_{m,k}, k)$$

continued

$$H_k = f(x_{1,k}, \ldots, x_{n,k}, u_{1,k}, \ldots, u_{m,k}, k)$$

$$+ \sum_{j=1}^{n} \lambda_{j,k+1} g_j(x_{1,k}, \ldots, x_{n,k}, u_{1,k}, \ldots, u_{m,k}, k)$$

$$\lambda_{j,k} = \frac{\partial H_k}{\partial x_{j,k}}$$

$$\lambda_{j,T} = \frac{\partial \phi}{\partial x_{j,T}}(x_{1,T}, \ldots, x_{n,T})$$

$$\frac{\partial H_k}{\partial u_{i,k}} = 0 \text{ at } (u^*_{1,k}, \ldots, u^*_{m,k})$$

Simple pest control problem

x good population and y pest population to be controlled with u.

$$\max_u \sum_{k=0}^{T} \frac{1}{2}\left[x_k - u_k^2\right]$$

$$x_{k+1} = x_k + x_k(1 - x_k) - x_k y_k \quad \text{for } k = 0, 1, \ldots, T,$$

$$y_{k+1} = y_k + x_k y_k - u_k y_k \quad \text{for } k = 0, 1, \ldots, T,$$

$$x_0 = 5, \; y_0 = 10,$$

$$0 \leq u_k \leq 1 \quad \text{for } k = 0, 1, \ldots, T.$$