Chapter 1

The Contributions of F.R. McMorris to Discrete Mathematics and its Applications

George F. Estabrook, Terry A. McKee, Henry Martyn Mulder, Robert C. Powers, Fred S. Roberts

In this chapter we discuss the contributions of F.R. McMorris to discrete mathematics and its applications on the occasion of his retirement in 2008.

Introduction

End of August 2008 F.R. McMorris, Buck for his friends and colleagues, retired as Dean of the College of Arts and Sciences at IIT, Illinois Institute of Technology, Chicago. He also celebrated his 65-th birthday. To commemorate these events a two-day conference was held early May 2008 at IIT. In addition this volume is written in honor of his contributions to mathematics and its applications. The focus of the volume is on areas to which he contributed most. The chapters show the broadness of his interests and his influence on many co-authors and other mathematicians. Here we survey his work. First some basic facts. At the moment of finishing this volume Math. Reviews lists 2 books, the editorship of 3 conference proceedings, and 108 published papers of Buck McMorris, and there are still many to come. There are 53 co-authors listed, and again there are more to come. Of course, the papers and co-authors listed in Math. Reviews are not all. The areas to which he has contributed, by number of publications, are: Combinatorics; Group theory and generalizations; Biology and other natural sciences; Game theory, economics, social and behavioral sciences; Operations research and mathematical programming; Order, lattices and ordered algebraic structures; Statistics; Computer science. Below we will highlight many of his contributions. Some characteristics of his way of working are: an open mind, keen on fundamental mathematics with a relevance for applications, always taking a broad view: try to formulate a 'master plan' that may be a guide for creating many specific questions and open problems. What seems to be equally important is that Buck has become a dear friend for many of his co-authors. A fact that was fundamental for the success, mathematically and socially, of the above mentioned celebration conference. Many of his co-authors are mentioned below, some are contributors of this volume. Much to our regret, because of the focus chosen, not all of his important co-authors are represented as author in this volume. Therefore we mention them here by number of their collaborations with Buck McMorris, according to MathSciNet: C.S. Johnson Jr., Bill Day, Frank Harary, Mike Jacobson, John Luedeman, Ewa Kubicka, Grzegorz Kubicki, Jean-Pierre Barthélémy, Robert Brigham, Hans-Hermann Bock, Jerald Kabell, and Chi Wang.

McMorris started his mathematical career in semi-groups. But even in his early work we can already discern some of his future interests in discrete mathematics and its applications: mathematical biology, intersection graphs, voting theory, consensus theory. In the early eighties his focus shifted from semigroups to discrete mathematics, with an emphasis on graph theory, while his early interests remained. In the sections below we highlight these. Of course, not all of his publications can be discussed. But the choices made provide a clear picture of his work and interests.

1.1. Mathematics of Evolutionary Biology

McMorris published 10 papers motivated by a concept from evolutionary systematic biology called character compatibility. Seven were between 1975 and 1981, the early days of his involvement with this concept: four of these included Estabrook as a coauthor [16; 17; 18; 19], two were entirely his own [41; 42] and one more with Zaslavsky [56]. McMorris et al. [55] addresses an abstract issue in graph theory related to an unresolved question from character compatibility, and Day et al. [13] apply McMorris' potential compatibility test to look for randomness in about 100 published data sets. For his last publication on character compatibility [20], he worked again with his original coauthor to examine the relationship between geologic stratigraphic data and compatibility.

To understand the relevance of McMorris' contributions to character compatibility analysis, it is useful to understand some of the concepts of evolutionary systematic biology. This subfield of biology seeks to estimate the tree of ancestor-descendant relationships among species, consequent of their evolution, and then use these evolutionary relationships to recognize higher taxa (groups of species in genera, families, etc). In the late 19th century, systematic biologists realized that similarities and differences with respect to a basis for comparison among a group of related species under study could be the basis for an hypothesis about the relationships among species and the ancestors from which they evolved, their so-called ancestor relation. Such hypotheses are expressed as characters, which group species together into the same character state if they are considered to be the same with respect to a basis for comparison, and then arrange these character states into a character state tree to indicate where speciation events associated with the observed changes are hypothesized to have occurred. By mid 20th century, some natural scientists also realized that some pairs of such hypotheses based on different bases for comparison could be logically incompatible, i.e., they could not both be true. At that time, scientists began to develop tests for, and ways to resolve, incompatibilities to estimate the ancestor relation from these hypotheses. Wilson (1965) [66] is among the earliest published works to present an explicit test for the compatibility of (two state) characters. Estabrook (2008) [21] provides an in-depth discussion of biological concepts of character state change, and the nature of compatibilities and incompatibilities among characters that arise from them. Estabrook (this volume) provides explicit explanations of McMorris' contributions motivated by this concept. Here I will summarize briefly what I consider to be his most significant contributions.

McMorris recognized a bi-unique correspondence between character state trees for a collection of related species and trees of subsets (ordered by inclusion) of that collection, which enabled a simple test for compatibility that identified the states involved with contradictions when the test failed. He realized that character state trees themselves enjoy a lower semi lattice order under the relation "is a refinement of", and described a simple test to recognize when a pair of character state trees were in that relation. Qualitative taxonomic characters are characters with their character states, but no explicitly hypothesized character state tree. Two qualitative taxonomic characters are potentially compatible if there exists character state trees for each that are compatible with each other. Estabrook had conjectured a simple test for potential compatibility, see [16; 6], which McMorris proved to be correct. Potential compatibility raises an unresolved issue: Several qualitative taxonomic characters can be pairwise potentially compatible but in some cases character state trees for each do not exist so that they remain pairwise compatible as character state trees. Simple criteria to recognize such cases have not yet been discovered. This is related to chordal graphs [24; 55]. McMorris' last publication addresses stratigraphic compatibility [20] and raises questions related to functional graphs. For an in-depth treatment of the papers discussed here, the reader is referred to Estabrook (this volume).

1.2. Contributions to Intersection Graph Theory

As in many parts of discrete mathematics, McMorris introduced or popularized significant new ideas in intersection graph theory, sometimes with a conference talk proclaiming a "master plan" for developing the idea. Six of these contributions are described below, with further discussion of each available in the 1999 SIAM monograph *Topics in Intersection Graph Theory* [40].

Upper bound graphs [40, §4.4]. The 1982 McMorris & Zaslavsky paper [57] combines McMorris's interests in partially ordered sets and graph representations. The *upper bound graph* G of a partial ordering $(P, <)$ has vertex set P, with distinct vertices adjacent in G if and only if the corresponding elements of P have a common upper bound in P. Reference [57] characterizes upper bound graphs by the existence complete subgraphs $Q_1, \ldots, Q_k$ that cover $E(G)$ such that, for each $j \leq k$, there exists a vertex v_j in G where $v_j \in Q_j$ but $v_j \notin Q_i$ for $i \neq j$; moreover, each Q_i can be assumed to be an inclusion-maximal complete subgraph of G. This characterization has spawned many related results, often coupled to applicable topics such as competition graphs.

Bipartite intersection graphs [40, §7.2]. The 1982 Harary, Kabell & McMorris paper [26] generalizes classical intersection graphs, and interval graphs in particular. A *bipartite intersection graph* G has $V(G)$ partitioned into sets X and Y, with each $x \in X$ and $y \in Y$ assigned sets S_x and T_y such that vertices x and y are adjacent in G if and only if $S_x \cap T_y \neq \emptyset$; furthermore, G is a *bipartite interval graph* if each S_x and T_y is an interval of the real line. Others have subsequently introduced concepts of directed intersection and interval graphs that are structurally interconnected with their bipartite counterparts.

***p*-Intersection graphs [40, §6.1].** The 1991 Jacobson, McMorris & Scheinerman paper [33] generalizes standard (1-)intersection graphs, significantly generalizing traditional intersection graph theory. The *p-intersection graph* G of a multiset $\{S_1, \ldots, S_k\}$ of subsets of an underlying finite set $\mathcal{X}$ has vertices $v_1, \ldots, v_k$, with distinct vertices v_i and v_j adjacent in G if and only if $|S_i \cap S_j| \geq p$. In particular, this extends the well-studied (but notoriously hard) concept of the *intersection number* of a graph—the minimum cardinality of $\mathcal{X}$ such that G is an intersection graph of subsets of $\mathcal{X}$—to *p-intersection numbers*.

Tolerance intersection graphs [40, §6.3]. The 1991 papers by Jacobson, McMorris & Mulder [32] and Jacobson, McMorris & Scheinerman [33] introduce this very general concept. The *ϕ-tolerance intersection graph* G of a family $\{S_1, \ldots, S_k\}$ of subsets of an underlying finite set $\mathcal{X}$ has each subset of S assigned a *measure* $\mu(S)$, has each S_i assigned a *tolerance* t_i, and has a binary function $\phi(x, y)$ that is often $\min\{x, y\}$, with distinct vertices v_i and v_j adjacent in G if and only if $\mu(S_i \cap S_j) \geq \phi(t_i, t_j)$. Tolerance intersection graphs generalize both p-intersection graphs and the previously-studied "tolerance graphs," which can now be described as interval graphs with μ the length of an interval and $\phi(x, y) = \min\{x, y\}$.

Sphere-of-influence graphs [40, §7.11]. The 1993 Harary, Jacobson, Lipman & McMorris paper [25] promotes ideas that were motivated by pattern recognition and computer vision problems. Suppose $\mathcal{X}$ is any finite set of points in the plane and each $x \in \mathcal{X}$ is associated with the open disc centered at x with radius equal to the minimum distance from x to the other points of $\mathcal{X}$. A *sphere-of-influence graph* G has vertices that correspond to such open discs, with distinct vertices adjacent in G if and only if the corresponding open discs have nonempty intersection. One basic question from [25] is which complete graphs are sphere-of-influence graphs—K_8 is; K_9 is conjectured to be; K_{12} is not. Closed sphere-of-influence graphs and ϕ-tolerance sphere-of-influence graphs have also been studied.

Probe interval graphs [40, §3.4.1] The 1998 McMorris, Wang & Zhang paper [54] developed tools that were directly motivated by work in physical mapping of DNA. A graph G is a *probe interval graph* if $V(G)$ contains a subset P and each vertex corresponds to an interval of the real line, with distinct vertices adjacent in G if and only if at least one of them is in P and their corresponding intervals have nonempty intersection. Reference [54] contributes structural information about probe interval graphs and has led to considerable recent work in this active research area.

1.3. Competition Graphs and their Generalizations

Buck McMorris has made some very interesting contributions to the study of competition graphs and the related phylogeny graphs. These topics combine his interests in graph theory with his interests in biology. This work was done in collaboration with Roberts (3 papers) and others.

1.3.1. *Competition Graph Definitions and Applications*

The study of competition graphs has given rise to a very large literature, some of which is surveyed in the articles [34; 37; 63]. Suppose $D = (V, A)$ is a digraph. Its *competition graph* $C(D)$ is the graph $G = (V, E)$ with the same vertex set and an edge $\{x, y\}$ in E for $x \neq y$ if and only if there is a vertex a in V so that arcs (x, a) and (y, a) are in D. Competition graphs were introduced by Cohen [9] in connection with a problem of ecology. The vertices of D represent species in an ecosystem and there is an arc from u to v if u preys on v. We call such a digraph a *food web*. There is an edge between species x and y in $C(D)$ if and only if x and y have a *common prey* a in D, *i.e.*, if and only if x and y compete for a. In the literature of competition graphs, it is very common to study the special case where D is an acyclic digraph without loops, as is commonly the case for food webs.

A variant of the competition graph idea is called the phylogeny graph because it was motivated by a problem in phylogenetic tree reconstruction. We say that G is the *phylogeny graph* $P(D)$ of $D = (V, A)$ if $G = (V, E)$ and there is an edge between $x \neq y$ in E if and only if either (x, a) and (y, a) are in A for some a in V, or (x, y) is in A or (y, x) is in A. If D is a digraph without loops and D' is the corresponding digraph with a loop added to each vertex, then it is easy to see that the phylogeny graph of D is the competition graph of D'. This observation was first made by Buck McMorris in a personal communication to Roberts. Roberts and Sheng [64] introduced the term phylogeny graph because of a possible connection of this concept to the problem of phylogenetic tree reconstruction. It is appropriate that Buck should have played a role in this notion of phylogeny graph because of his longstanding interest and many contributions to the theory and practice of phylogenetic tree reconstruction.

The notion of competition graph also arises in a variety of other non-biological contexts. (See [61].) Suppose the vertex set of D can be divided into two classes, A and B, and all arcs are from vertices of A to vertices of B. (We do not assume that A and B are disjoint.) Then we sometimes seek the restriction of the competition graph to the set A. This idea arises for instance in communications where A is a set of transmitters, B is a set of receivers, and there is an arc from u in A to v in B if a message sent at u can be received at v. We then note that x and y in A *interfere* if signals sent at x and y can be received at the same place, *i.e.*, if and only if x and y are adjacent in the competition graph (restricted to A). The problem of channel assignment in communications can be looked at as the problem of coloring the *interference graph*.

The idea also arises in coding. Suppose A is a transmission alphabet, B is a receiving alphabet, and there is an arc from u in A to v in B if when symbol u is sent, symbol v can be received. Then symbols x and y in the transmission alphabet are *confusable* if they can be received as the same letter, *i.e.*, if and only if x and y are adjacent in the competition graph (restricted to A). We often seek a minimum set of mutually non-confusable symbols in a transmission alphabet – this is the problem of finding a maximum independent set in the competition graph (restricted to A).

Competition graphs arise in scheduling in situations where we have conflicting requests. Suppose that A is the set of users of a facility and B the set of facilities, and an arc from u in A to v in B means that user u wishes to use facility v. Then users x and y *conflict* if they both wish to use the same facility. In another scheduling application, A is a set of users of a fixed facility and B the set of times that facility might be used, and an arc from u in A to v in B means that user u wishes to use the facility at time v. Users x and y conflict if they both wish to use the facility at the same time. The competition graph is sometimes called the *conflict graph*.

Competition graphs arise in studies of the structure of models of complex systems arising in modeling of energy and economic systems. In such models, we often use matrices and set up linear programs. Let A be the set of rows of a matrix M and B the set of columns, and take an arc from u to v if the u, v entry of M is nonzero. Then in a corresponding linear program, the constraints corresponding to rows x and y involve a common variable with nonzero coefficients if and only if x and y are adjacent in the competition graph. In the literature, the competition graph is called the **row graph** of matrix M. The row graph is useful in understanding the structure of linear programs.

1.3.2. *Competition Numbers and Phylogeny Numbers*

As noted earlier, Buck McMorris has made extensive contributions to the theory and applications of interval graphs. Interval graphs have played a central role at the interface between mathematics and biology, and the connection between interval graphs and competition graphs has been a primary force in leading to the great interest in competition graphs. In ecology, a species' normal healthy environment is characterized by allowable ranges of different important factors such as temperature, humidity, pH, etc. If there are p factors and each is taken to be a dimension in Euclidean p-space, then if the ranges on the different factors are independent (a simplifying assumption), the species can be represented by a box in p-space. This box is called the species' *ecological niche.* An old ecological principle says that two species compete if and only if their ecological niches overlap. (That is why the competition graph is sometimes called the *niche overlap graph.*) Cohen [9; 10; 11] asked if, given an independent notion of competition, we could assign each species in an ecosystem to an ecological niche in such a way that competition between species corresponds to overlap of niches. In particular, he started with a food web or digraph with an arc from u to v if u preys on v, defined the corresponding competition graph, and asked if the competition graph could be represented as the intersection graph of boxes in p-space. More specifically, he asked for the smallest such p, which is known as the *boxicity* of the competition graph. Cohen [9] made the remarkable observation that in a large number of examples of food webs, the boxicity of the competition graph always turned out to be 1, i.e., that the competition graph was always an interval graph. In other words, only one ecological dimension sufficed to account for competition. The interpretation of this dimension was (and is) unclear. Although later examples were found by Cohen and others to show that not every competition graph had boxicity 1, Cohen's original observation and the continued preponderance of examples with boxicity 1 led to a large literature devoted to attempts to explain the observation and to study the properties of competition graphs.

In attempting to explain the observation that most real world food webs have competition graphs that are interval graphs, Roberts [62] asked whether this was just a property of the construction, *i.e.*, whether most acyclic digraphs have competition graphs that are interval graphs. He noted that if G is any graph, then G plus sufficiently many isolated vertices is a competition graph of an acyclic digraph. Roberts then defined the *competition number* $k(G)$ of a graph G as the smallest r so that G plus r isolated vertices is a competition graph of an acyclic digraph. Thus, any algorithm for recognizing competition graphs of acyclic digraphs will also compute the competition number, and conversely.

1.3.3. *p-Competition Graphs*

A large number of variations of the notion of competition graph have given rise to interesting problems and questions. To define one such variation, suppose $D = (V, A)$ is a digraph. The *p-competition graph* of D has vertex set V and an edge between x and y in V if there are distinct vertices a_1, a_2, ..., a_p in V so that (x, a_i) and (y, a_i) are arcs in D for $i = 1, 2, ..., p$. In terms of the ecological motivation, x and y compete if and only if they have at least p common prey. This idea was studied by Buck McMorris and collaborators in a series of three papers: [30; 35; 36]. A variety of results analogous to those about ordinary competition graphs are known. Paper [36] by McMorris and coauthors gave necessary and sufficient conditions for a graph with n vertices to be the p-competition graph of some acyclic digraph.

It also provides similar results for arbitrary digraphs (loops allowed) and arbitrary digraphs (loops not allowed). Graph-theoretically, the most interesting results arise if one studies p-competition graphs of arbitrary digraphs. So far, most of the interesting results are about the case $p = 2$. Paper [36] showed that every triangulated graph is a 2-competition graph of an arbitrary digraph. So is every unicyclic graph except the 4-cycle.

The question of what complete bipartite graphs $K(m, x)$ are 2-competition graphs of arbitrary digraphs leads to some very interesting (and difficult) combinatorial questions. In paper [30], McMorris and colleagues showed that $K(2, x)$ is a 2-competition graph of an arbitrary digraph if and only if $x = 1$ or $x \geq 9$ and that $K(3, x)$ is not a 2-competition graph of an arbitrary digraph if $x = 3$, 4, 5, 7, 8, 11. Then, Jacobson [31] showed that $K(3, x)$ is a 2-competition graph of an arbitrary digraph for $x \geq 38$. The situation for $K(3, 6)$ and $K(3, 37)$ remains open, to our knowledge.

Also of interest is a concept analogous to competition number. The *$p-$competition number* $k_p(G)$ is the smallest r so that G together with r isolated vertices is a p-competition graph of some acyclic digraph. McMorris and his colleagues [35] showed that this is well-defined. In this same paper, they showed the surprising result that for every m, there is a graph G with $k_p(G) \leq k(G) - m$.

1.3.4. *Tolerance Competition Graphs*

As noted above in Section 1.2, the 1991 papers by Jacobson, McMorris and Mulder [32] and Jacobson, McMorris and Scheinerman [33] introduced a very general concept called a ϕ-tolerance intersection graph. An analogous notion for competition graphs was introduced by Brigham, McMorris, and Vitray [7; 8]. Let ϕ be a symmetric function assigning to each ordered pair of natural numbers another natural number. We say that $G = (V, E)$ is a *ϕ-tolerance competition graph* if there is a directed graph $D = (V, A)$ and an assignment of a nonnegative integer t_i to each vertex v_i in V such that, for $i \neq j$,

$$\{v_i, v_j\} \in E(G) \leftrightarrow |\{a : (v_i, a) \in A\} \cap \{a : (v_j, a) \in A\}| \geq \phi(t_i, t_j).$$

A *2-ϕ-tolerance competition graph* is a ϕ-tolerance competition graph in which all the t_i are selected from a 2-element set. Characterizations of such graphs, and relationships between them, are presented for ϕ equal to the minimum, maximum, and sum functions, with emphasis on the situation in which the 2-element set is $\{0, q\}$.

1.4. Location Functions on Graphs

As mentioned in Section 1.2 Buck McMorris used the idea of a "master plan" to generate all kinds of interesting questions and problems. This inspired his coauthor Mulder to use this meta-concept as well. The first instance was Mulder's "Meta-conjecture" mentioned in Mulder (this volume). Trees and hypercubes share being median graphs. In a sense they are the extreme cases within this class, in the class of all median graphs with n vertices, the trees realize the minimum number of edges: $n-1$, and the n-cube Q_n realizes the maximum number of edges 2^n. The following has served as a "master plan" in the sense of McMorris.

Metaconjecture. *Let $\mathcal{P}$ be a property that makes sense, which is shared by the trees and the hypercubes. Then $\mathcal{P}$ is shared by all median graphs.*

The reader is referred to Mulder (this volume) for the incentive this Metaconjecture has given. In the spirit of this Metaconjecture one also tries to generalize results on trees to median graphs whenever possible. This was the motivation for Buck McMorris to study the axiomatic characterization of locations functions on median graphs. Location functions are are a specific instance of consensus functions. A consensus function is a model to describe a rational way to obtain consensus among a group of agents or clients. The input of the function consists of certain information about the agents, and the output concerns the issue about which consensus should be reached. The rationality of the process is guaranteed by the fact that the consensus function satisfies certain "rational" rules or "consensus axioms". For a location function on a network the input is the position of the clients in the network, and the output is the set of preferred locations. For a full discussion of the axiomatic characterizations of three important location functions see McMorris, Mulder, and Vohra (this volume), where the details of the results discussed below can be found.

A central problem in location theory and consensus theory is to find those points in a set X that are "closest" to any given profile $\pi = (x_1, x_2, \ldots, x_k)$. Most of the work done in this area focuses on developing algorithms to find these points [12; 58]. In recent years, there have been axiomatic studies of the procedures themselves and these have resulted in a much better understanding of the process of location and consensus [4; 5; 23; 27; 28]. Without any conditions imposed, a *location function* (*consensus function*) on X is simply a mapping $L : X^* \to 2^X - \emptyset$, where X^* is the set of all profiles of all finite lengths and $2^X - \emptyset$ denotes the set of all nonempty subsets of X.

Let $\delta : X \times X^* \to \mathbb{R}$ be a function such that $\delta(x, \pi)$ represents a measure of "remoteness" of x to the profile π. An attractive class of location functions on (X, δ) is defined by letting $L(\pi) = \{x \in X : \delta(x, \pi) \text{ is minimum}\}$. Two important location functions in this class are the *median function* Med, defined by letting $\delta(x, \pi) = \sum_{i=1}^{k} \delta(x, x_i)$, where $\pi = x_1, x_2, \ldots, x_k$, and the *center function* Cen, defined by letting $\delta(x, \pi) = \max\{\delta(x, x_1), \delta(x, x_2), \ldots, \delta(x, x_k)\}$.

In the continuous case we consider connected networks $N = (V, A)$ with vertex set V and set of arcs A. Think of N as being embedded in n-space. The arcs are curves with a length. The set X is the set of all vertices *and* all interiors points on the arcs. In the discrete case we consider connected graphs $G = (V, E)$ with vertex set V and edge set E. Now the set X is the set of vertices V. Note that there might be big differences between the continuous and the discrete case. For instance, the center function Cen is single-valued in the continuous case but not in the discrete case. Also proof techniques may be quite different.

In 1996 Vohra [65] characterized the median function on tree networks axiomatically, where only three simple axioms were needed. In a tree network the set X is the set of all vertices *and* interior points on the arcs, where arcs can have any length. This is the "continuous case". Rephrased Vohra's axioms are as follows. the *segment* $S(x, y)$ between x and y in a tree network is the set of all point on the path between points x and y. For two profiles π and ρ we denote the concatenation of these by π, ρ.

(A) Anonymity: for any profile $\pi = x_1, x_2, \ldots, x_k$ on X and any permutation σ of $\{1, 2, \ldots, k\}$, we have $L(\pi) = L(\pi^\sigma)$, where $\pi^\sigma = x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)}$.

(B) Betweenness: [Continuous] $L(x, y) = S(x, y)$, for all $x, y \in X$.

(C) Consistency: If $L(\pi) \cap L(\rho) \neq \emptyset$ for profiles π and ρ, then $L(\pi, \rho) = L(\pi) \cap L(\rho)$.

Note that it is easy to show that Med satisfies these three axioms. But Vohra proved the 'converse' as well: any consensus function on a tree network satisfying (A), (B), and (C) necessarily is the median function on the tree network.

When McMorris started to work on the discrete case for Med he realized that it should be done on median graphs. Now the betweenness axiom has to be adapted to the discrete case. The *interval* $I(u, v)$ between vertices x and y in a graph $G = (V, E)$ is the set of vertices lying on the shortest paths between x and y.

(B) Betweenness: [Discrete] $L(u, v) = I(u, v)$, for all $u, v \in V$.

In [46] it was proved that the median function Med on cube-free median graphs is characterized by the three obvious axioms (A), (B), and (C). A median graph G is *cube-free* if G does not contain a 3-cube Q_3. Such graphs are a nice generalization of trees. They seem to be quite esoteric, but there is a one-to-one correspondence between the class of connected triangle-free graphs and a subclass of the cube-free median graphs, see [29]. For the class of all median graphs McMorris, Mulder and Roberts [46] needed an extra 'heavy duty' axiom. These results were extended in [43; 44], where also the ordered case, viz. distributive and median semilattices, is discussed. Another interesting case initiated by McMorris is the *t-Median Function*, see [45]. We omit details.

There is not as much known for Cen on graphs. Foster and Vohra [22] studied the center function on tree networks. A breakthrough occurred when Buck McMorris and coauthors [53] succeeded in characterizing the center function on trees as we have defined it. The result is that a location function L on a tree T is the center function Cen if and only if L satisfies the following four axioms. For a profile π we denote the set of vertices in π by $\{\pi\}$. For a vertex x we denote by $\pi \setminus x$ the subprofile of π by deleting all occurrences of x from π. For a profile π on a tree T we denote by $T(\pi)$ the smallest subtree of T containing all of π.

(Mid) Middleness: [Discrete] Let u, v be two not necessarily distinct vertices of a tree T. Then $L(u, v)$ is the middle of the unique path joining u and v in T.

(QC) Quasi-consistency: If $L(\pi) = L(\rho)$ for profiles π and ρ, then $L(\pi, \rho) \;=\; L(\pi)$.

(R) Redundancy: Let L be a location function on a tree T. If $x \in T(\pi \setminus x)$ then $L(\pi \setminus x) = L(\pi)$.

(PI) Population Invariance: If $\{\pi\} = \{\rho\}$ then $L(\pi) = L(\rho)$.

A shorter proof if this result can be found in [60]. A closer look at that proof yields that an analogous result holds for the continuous case, i.e., for tree networks.

McMorris and his coworkers are still continuing research on these location functions, but also other nice instances as the *Mean Function*. A *mean vertex* of π is a vertex v minimizing

$$\sum_{1 \le i \le k} [d(v, x_i)]^2.$$

The *mean* of π is the set of mean vertices of π. The *Mean Function* $Mean$ on G is the function $Mean : VX^* :\rightarrow 2^V - \emptyset$ with $Mean(\pi)$ being the mean of π.

1.5. Contributions to Bioconsensus: An Axiomatic Appraoch

Buck McMorris has made many contributions to the area of mathematical consensus and a few of these contributions will be mentioned in this section. We first describe what is meant by a consensus function and then we introduce two well known axioms a given consensus function may or may not satisfy.

Let $\mathcal{D}$ be the set of all (finite) discrete structures of a particular type. (e.g., $\mathcal{D}$ could be a set of specialized labelled graphs, unlabelled graphs, digraphs, partially ordered sets, acyclic digraphs, hypergraphs, partitions, networks, etc.) A *consensus function* on $\mathcal{D}$ is a map $C : \mathcal{D}^k \longrightarrow \mathcal{D}$, where $k \geq 2$ is a positive integer. A major aspect of the *consensus problem* for $\mathcal{D}$ is to find "good" consensus functions that can capture the common agreement of an input *profile* $P = (D_1, \ldots, D_k)$ of members of $\mathcal{D}$, i.e. $C(P)$ should consist of the element (or elements) of $\mathcal{D}$ that best represents whatever similarity that all of the D_i's share. If possible, a good function C should not only have this "consensus" aspect, but additionally should satisfy mathematical properties that enable it to be understood in order that it can be effectively computed exactly or with approximating algorithms. The consensus problem for discrete structures has been a very active area of research with much of it stimulated by the axiomatic approach to social choice (voting theory) pioneered by K. Arrow in the 1950's. In the classical theory developed by Arrow and others, $\mathcal{D}$ is usually taken to be the set of all weak or linear orders on a given set of alternatives S. Many of the axioms are given in terms of the "units of information" (building blocks) of members of $\mathcal{D}$, which in the case for partial orders, are the ordered pairs of S making up the order relation. (Other discrete structures obviously have other types of building blocks.) For example, in generic terms, a property that is universally accepted as being desirable for data aggregation is the following: A consensus function $C : \mathcal{D}^k \longrightarrow \mathcal{D}$ is *Pareto* (P) if whenever $P = (D_1, \ldots, D_k)$ is a profile and 'unit of information' x is in every D_i, then x is in $C(P)$. The Pareto condition simply requires the preservation of the unanimous agreement portion of the input data profile. Another seemingly reasonable property is the following: A consensus function C is *independent (of irrelevant alternatives)* (I) if whenever profiles P and P' agree on a subset $X \subseteq S$, then $C(P)$ and $C(P')$ agree on X. This independence condition also seems to be a good one and captures an aspect of a "stable" consensus function. Of course, what it means to "agree" must be carefully defined. When $\mathcal{D}$ is the set of all weak orders on S (reflexive, transitive and complete relations on S), agreement of two weak orders simply means that they are equal as sets of ordered pairs when restricted to elements in X. Profiles then are said to agree on X if they agree term by term on X. The famous Impossibility Theorem of Arrow essentially says that the only consensus functions on weak orders (where $|S| \geq 3$) satisfying both (P) and (I) are the *dictatorships*, i.e., there is an index j such that for any profile P, if x is strictly preferred to y in D_j, then x is strictly preferred to y in $C(P)$ [1].

Buck McMorris , along with his coauthors, has extended Arrow's Impossibility Theorem in many different directions. For example, in 1983, McMorris and Neumann proved an analog of Arrow's Theorem for tree quasi-orders [47]. A tree quasi-order is a binary relation ρ on a finite set S such that ρ is reflexive, transitive, and $(z, x), (z, y) \in \rho$ implies that $(x, y) \in \rho$ or $(y, x) \in \rho$ for all $x, y, z \in S$. The last condition is called the *tree condition* and it is a generalization of completeness. In 2004, McMorris and Powers extended the tree quasi-order version of Arrow's Theorem by dropping the Pareto condition and replacing it with two profile conditions [49]. In this case, it was shown that the resulting class of consensus functions are quasi-oligarchic. In 1991, Barthelemy, McMorris and Powers, using a carefully constructed independence axiom, established a version of Arrow's Theorem for consensus functions defined on the set $\mathcal{H}(S)$ of all hierarchies of S [2]. A *hierarchy* on S is a collection H of subsets of S such that $S, \{x\} \in H$ for all $x \in S$; $\emptyset \notin H$; and $A \cap B \in \{A, B, \emptyset\}$ for all $A, B \in H$. In 1995, Barthelemy, McMorris and Powers investigated eight different versions of independence conditions for consensus functions on $\mathcal{H}(S)$ and the complete relationships among these eight conditions were determined [3]. In 2003, a deeper connection was made between consensus functions on weak orders and consensus functions on $\mathcal{H}(S)$ with the possibility of having an infinite number of voters. A key to this connection is to view a hierarchy as a special type of ternary relation. Using this viewpoint, it was shown how to embed and extend Arrow's Theorem for weak orders to a result involving ternary relations [48].

In 1952, Kenneth May gave an elegant characterization of simple majority decision based on a set with exactly two alternatives [39]. This work is a model of the classical voting situation where there are two candidates and the candidate with the most votes is declared the winner. May's theorem is a fundamental result in the area of social choice and it has inspired many extensions. In particular, in 2008, McMorris and Powers generalized May's Theorem to the case of three alternatives, but where the voters' preference relations are required to be trees with the alternatives at the leaves [51].

A popular consensus function on the set of hierarchies $\mathcal{H}(S)$ is *majority rule*. Majority rule outputs the set of clusters that appear in more than half of the input profile. The fact that the output is a hierarchy was first noted in 1981 by Margush and McMorris [38]. Although this result is easy to prove, it stands in stark contrast to the situation in classical voting theory where the majority outcome could produce what is called a *voting paradox*. In 2008, McMorris and Powers proved that the majority consensus rule on hierarchies is the only consensus function satisfying four natural and easily stated axioms [50]. The majority consensus rule is part of a larger class of consensus rules where the output is determined by a family of overlapping sets, often called *decisive sets*. Axiomatic characterizations of this class of consensus rules can be found in [47] and [52]. Finally, for a more thorough discussion of axiomatic consensus theory we refer the reader to the book by Day and McMorris [14].

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